

Riemann Surfaces, Examples 3

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1. Suppose $\Omega \subset \mathbb{C}$ is an additive subgroup such that Ω contains only isolated points. Show that *either* $\Omega = \{0\}$, *or* $\Omega = \mathbb{Z}\omega$ for some $\omega \neq 0$, *or* $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1, \omega_2 \neq 0$ and $\omega_2/\omega_1 \notin \mathbb{R}$.
2. Suppose that f is a simply periodic analytic function on \mathbb{C} with periods \mathbb{Z} , and that $\lim_{y \rightarrow +\infty} f(x + iy)$ and $\lim_{y \rightarrow -\infty} f(x + iy)$ both exist (possibly ∞) uniformly in x . Show that $f(z) = \sum_{k=-n}^n a_k e^{2\pi i k z}$, i.e. $f(z)$ has a *finite* Fourier expansion.
3. Let f be a non-constant elliptic function with respect to a lattice $\Lambda \subset \mathbb{C}$. Let $P \subset \mathbb{C}$ be a fundamental parallelogram; using the argument principle, and if necessary slightly perturbing P , show that the number of zeros of f in P is the same as the number of poles, both counted with multiplicities (in lectures, this followed by a use of the Valency theorem, but this more direct argument via contour integration also works).
4. With the notation as in the previous question, let the degree of f be n , and let a_1, \dots, a_n denote the zeros of f in a fundamental parallelogram P , and let b_1, \dots, b_n denote the poles (both with possible repeats). By considering the integral (if required, also slightly perturbing P)

$$\frac{1}{2\pi i} \int_{\partial P} z \frac{f'(z)}{f(z)} dz,$$

show that

$$\sum_{j=1}^n a_j - \sum_{j=1}^n b_j \in \Lambda.$$

5. Suppose a is a complex number with $|a| > 1$. Show that any analytic function f on \mathbb{C}_* with $f(az) = f(z)$ for all $z \in \mathbb{C}_*$ must be constant, but that there is a non-constant meromorphic function f on \mathbb{C}_* with $f(az) = f(z)$ for all $z \in \mathbb{C}_*$.
6. Let $\wp(z)$ denote the Weierstrass \wp -function with respect to a lattice $\Lambda \subset \mathbb{C}$. Show that \wp satisfies the differential equation $\wp''(z) = 6\wp(z)^2 + A$, for some constant $A \in \mathbb{C}$. Show that there are at least three points and at most five points (modulo Λ) at which \wp' is not locally injective.

7. With notation as in the previous question, and a a complex number with $2a \notin \Lambda$, show that the elliptic function

$$h(z) = (\wp(z-a) - \wp(z+a))(\wp(z) - \wp(a))^2 - \wp'(z)\wp'(a)$$

has no poles on $\mathbb{C} \setminus \Lambda$. By considering the behaviour of h at $z = 0$, deduce that h is constant, and show that this constant is zero.

8. Find an explicit regular covering map of Riemann surfaces $\Delta \rightarrow \Delta^*$, where Δ here denotes the open unit disc and Δ^* the punctured disc.
9. Show that $\mathbb{C} \setminus \{P, Q\}$, where $P \neq Q$, is not conformally equivalent to \mathbb{C} or \mathbb{C}_* , and deduce from the Uniformization theorem that it is uniformized by the open unit disc Δ . Show that the same is true for any domain in \mathbb{C} whose complement has more than one point.
10. Let R be a compact Riemann surface of genus g and P_1, \dots, P_n be distinct points of R . Show that $R \setminus \{P_1, \dots, P_n\}$ is uniformized by the open unit disc Δ if and only if $2g - 2 + n > 0$, and by \mathbb{C} if and only if $2g - 2 + n = 0$ or -1 .
11. Let f, g be meromorphic functions on a compact Riemann surface R . Show that there is a non-zero polynomial $P(w_1, w_2)$ such that $P(f, g) = 0$.
[Hint: Suppose f, g have valencies m, n respectively, and put $d = m + n$. Show that it is possible to choose complex numbers a_{ij} , not all zero, such that the function

$$\sum_{j=0}^d \sum_{k=0}^d a_{jk} f(z)^j g(z)^k$$

has at least $(d^2 + 2d)$ distinct zeros in R . Show that it cannot have more than d^2 poles, and deduce that it must be identically zero on R .]

12. Prove from first principles that S^2 is simply connected.
13. Fix $d \geq 2$ and let $F'_d := \{(x, y) \in \mathbb{C}^2 : x^d + y^d = 1\}$. Show that the coordinate projections give F'_d the structure of a Riemann surface, and use topological gluing to find a compact Riemann surface F_d into which F'_d analytically embeds. Prove that the coordinate maps extend to meromorphic functions on F_d .